

# CONVEX DOMAINS WITH LOCALLY LEVI-FLAT BOUNDARIES

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**ABSTRACT.** It is shown that a domain in  $\mathbb{C}^n$  which is locally convex and has  $\mathcal{C}^1$ -smooth Levi-flat boundary is locally linearly equivalent to a Cartesian product of a planar domain and  $\mathbb{C}^{n-1}$ .

A hypersurface  $M \subset \mathbb{C}^n$  is called Levi-flat near a  $\mathcal{C}^1$ -smooth point  $p \in M$  if  $M$  divides  $\mathbb{C}^n$  near  $p$  into two pseudoconvex domains (say  $M^+$  and  $M^-$ ). By [1],  $M$  is Levi-flat near  $p$  if and only if  $M$  admits a (unique)  $\mathcal{C}^1$ -smooth foliation near  $p$  by one-codimensional complex manifolds. It is well-known that if  $M$  is real-analytic, then it is locally biholomorphic to a real hyperplane.

We have more in the  $\mathcal{C}^1$ -smooth “convex” case.

**Proposition 1.** *Let  $M \subset \mathbb{C}^n$  be a Levi-flat hypersurface near a  $\mathcal{C}^1$ -smooth point  $p \in M$  such that  $M^+$  is convex. Then  $M$  near  $p$  is linearly equivalent to the Cartesian product of a planar curve and  $\mathbb{C}^{n-1}$ .*

*Proof.* After an affine change of the variables, we may assume that  $p = 0$  and the (real) tangent hyperplane to  $M$  at 0 is  $\{\operatorname{Re} z_1 = 0\}$ . In what follows,  $\delta$  will stand for a positive constant which will be shrunk as needed in the course of the proof. By convexity  $T_\delta = \{z_1 = 0\} \cap \mathbb{D}^n(0, \delta) \subset \overline{M^-}$  ( $\mathbb{D}^n$  denotes a polydisc), and by  $\mathcal{C}^1$ -smoothness we may assume that  $T_\delta + (\varepsilon, 0, \dots, 0) \subset M^-$  for any small  $\varepsilon > 0$ . Since  $M^-$  is taut near 0 (because of  $\mathcal{C}^1$ -smoothness and pseudoconvexity), then  $T_\delta \subset M$ .

Hence  $M_\delta = M \cap \mathbb{D}^n(0, \delta)$  is foliated by affine complex hyperplanes. The set  $M_\delta \cap (\mathbb{C} \times \{(0, \dots, 0)\})$  is parametrized by  $-g(t) + it$ , where  $g$  is a convex, nonnegative  $\mathcal{C}^1$ -smooth function with  $g(0) = g'(0) = 0$ . Then there exists a continuous map  $\alpha : (-\delta, \delta) \mapsto \mathbb{C}^{n-1}$  so that  $M_\delta$  is parametrized by

$$(1) \quad (-g(t) + it + \alpha(t) \cdot z', z'), \quad z' = (z_2, \dots, z_n).$$

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Since  $M_\delta \subset \{\operatorname{Re} z_1 \leq 0\}$ , we have (shrinking  $\delta$  again)

$$-g(t) + \operatorname{Re}(\alpha(t) \cdot z') \leq 0 \text{ for any } z' \in \mathbb{D}^{n-1}(0, \delta).$$

This implies that  $\|\alpha(t)\| \leq g(t)/\delta$ .

Now consider more generally a point  $p_0 := (-g(t_0) + it_0, 0) \in M$ . In the new coordinates

$$\tilde{z}_1 = (1 - ig'(t_0)) [(z_1 + g(t_0) - it_0) - \alpha(t_0) \cdot z'], \quad \tilde{z}' = z',$$

one can check that the real tangent hyperplane to  $M$  at  $p_0$  is  $\{\operatorname{Re} \tilde{z}_1 = 0\}$ . The equation of the hyperplane given by (1) becomes

$$\tilde{z}_1 = (1 - ig'(t_0)) [(-g(t) + g(t_0) + i(t - t_0) + (\alpha(t) - \alpha(t_0)) \cdot \tilde{z}')] ,$$

and using the fact that  $M \subset \{\operatorname{Re} \tilde{z}_1 \leq 0\}$ , for  $\|\tilde{z}'\| \leq \delta_1$ ,

$$\begin{aligned} \sqrt{1 + g'(t_0)^2} \|\alpha(t) - \alpha(t_0)\| &\leq [g(t) - g(t_0) - (t - t_0)g'(t_0)] / \delta \\ &= (t - t_0)(g'(t_1) - g'(t_0)) / \delta \end{aligned}$$

for some value  $t_1$  between  $t_0$  and  $t$ , by the Mean Value Theorem. Using the uniform continuity of  $g'$ , we find some interval around 0, where for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that  $|t - t_0| \leq \eta$  implies  $\|\alpha(t) - \alpha(t_0)\|/|t - t_0| \leq \varepsilon$ . We deduce that  $\alpha$  must be constant over that interval which yields the product structure that was claimed.  $\square$

**Remarks.** 1. The argument at the beginning of the proof shows that if the boundary of a linearly convex domain is locally Levi-flat, then it too is foliated by complex hyperplanes. On the other hand, the conclusion of this proposition does not hold in the linearly convex case, even up to transformations preserving affine complex hyperplanes: the boundary of symmetrized bidisc  $\mathbb{G}_2 = \{z \in \mathbb{C}^2 : |z_1 - \bar{z}_1 z_2| + |z_2|^2 < 1\}$  locally contains the tangent line at any smooth point but  $\mathbb{G}_2$  is not locally fractional linearly equivalent to a Cartesian product.

2. The result does not extend to the case where the Levi form has positive constant rank (strictly less than the maximal rank  $n - 1$ ). For example, consider the tube hypersurface over the cone  $x_1 x_3 = x_2^2$  ( $x_1 > 0$ ), i.e. the hypersurface in  $\mathbb{C}^3$  given by

$$M := \{z \in \mathbb{C}^3 : \rho(z) := (\operatorname{Re} z_2)^2 - (\operatorname{Re} z_1)(\operatorname{Re} z_3) = 0, \operatorname{Re} z_1 > 0\}.$$

Then  $\{\rho < 0 < \operatorname{Re} z_1\}$  is convex, the Levi form of  $\rho$  is semi definite positive and of constant rank 1 on  $M$ , and  $M$  is foliated by portions of the affine complex lines

$$\{(a^2 \zeta + ib_1, a\zeta + ib_2, \zeta), \zeta \in \mathbb{C}\}, \quad a, b_1, b_2 \in \mathbb{R},$$

which of course are not all parallel to each other.

## REFERENCES

- [1] R. A. Airapetian, *Extending CR functions from piecewise smooth CR manifolds*, Math. Sbornik 134 (1987), 108–118 (in Russian).

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